NOVIKOV - SHUBIN SIGNATURES, I

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Abstract. Torsion objects of von Neumann categories describe the phenomenon "spectrum near zero" discovered by S. Novikov and M. Shubin. In this paper we classify Hermitian forms on torsion objects of a finite von Neumann category. We prove that any such Hermitian form can be represented as the discriminant form of a degenerate Hermitian form on a projective module. We also find the relation between the Hermitian forms on projective modules which holds if and only if their discriminant forms are congruent. A notion of superfinite von Neumann category is introduced. It is proven that the classification of torsion Hermitian forms in a superfinite category can be completely reduced to the isomorphism types of their positive and the negative parts.

§0. Introduction

S. Novikov and M. Shubin [NS1], [NS2] discovered a new way of producing topological invariants of manifolds by studying the spectrum near zero of the Laplacian acting on L^2 forms on the universal cover. It was proven by M. Gromov and M. Shubin [GS1], [GS2] that the Novikov - Shubin invariants depend only on the homotopy type of the manifold. W. Lück and J. Lott [LL] computed the Novikov - Shubin invariants for some 3-dimensional manifolds.

Two different homological "explanations" the Novikov - Shubin invariants were suggested independently by W. Lück [Lu1], [Lu2],[Lu3] and by the author [Fa1], [Fa2].

The main idea of the approach developed in [Fa1 - Fa2] was to view the "spectrum near zero" as *a torsion part of the L^2 cohomology, understood in an extended sense.* The main principles of [Fa1-Fa2] are the following:

- *the category of Hilbert representations of a given descrete group can be canonically embedded into an abelian category $\mathcal{E}$;*
- *the extended abelian category $\mathcal{E}$ contains the chain complex $\ell^2(\pi)\otimes_\pi C_*(\tilde M)$ of L^2 forms on the universal covering $\tilde M$ for any compact polyhedron M;*
- *the homology of this chain complex (called the extended L^2 cohomology) is a well defined object of $\mathcal{E}$.*
- *the extended cohomology naturally splits as a sum of its projective and torsion parts;*
- *the theory of von Neumann dimension provides a tool to measure the size of the projective part of the extended cohomology;*
- *the Novikov-Shubin invariants depend only of the torsion part of the extended cohomology.*

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There are numerous advantages of viewing the Novikov - Shubin invariants as byproducts of some homology theory. Firstly, it immediately implies their homotopy invariance (the result of Gromov and Shubin). Secondly, there exist other invariants of torsion part of the extended L^2-homology, (for example, the ones constructed in [Fa1], [Fa2] and others) which are independent of the Novikov - Shubin invariants. Thirdly, using some of these invariants allows strengthening (obtained in [Fa1], [Fa2]) of the Morse type inequalities of Novikov and Shubin [NS1], [NS2]. At last, as we showed in [Fa1], [Fa2] the well developed tools of homological algebra (such as derived functors and spectral sequences) can be used in order to compute the Novikov - Shubin invariants.

In [Fa3] we showed that the construction of the extended abelian category and extended L^2 cohomology applies in many cases when one deals with *finite von Neumann categories*. The category of Hilbert representations of a descrete group (considered in [Fa1], [Fa2]) is a specific example of finite von Neumann category. Other examples, studied in [Fa4], include description of the geometry of growth and the behavior of the Betti numbers under growth (Lück [L4] type theorems).

The main purpose of the present paper is to classify Hermitian forms on torsion objects of a finite von Neumann category. In the future publications we will apply the results obtained here to study the spectrum near zero of the Dirac operator acting on the universal covering (the Novikov - Shubin signatures); we will construct a combinatorial analogue of the eta-invariant; also, we will apply the technique of this paper to study the knot concordance problem (in particular we will get a version of the Casson - Gordon knot cobordism invariant).

The definition of a Hermitian form on torsion Hilbertian modules is not straightforward. Indeed, such torsion module is not a set, only an object of certain category. In this paper we adopt the general formalism of Hermitian forms in categories, as developed by H.-G. Quebbemann, W. Scharlau, M. Schulte [QSS], A. Ranicki [R1, R2], C.T.C. Wall [W1, W2] and others. We review this formalism in §1. In §3 a duality in the category of torsion Hilbertian modules is described; using the duality, one may apply the general formalism of Hermitian forms in this situation.

In section §4 we show that any non-degenerate torsion Hermitian form can be represented as a *discriminant form* of a *degenerate form* on a projective object, Theorem 4.7. Because of this, the study of non-degenerate torsion forms can be viewed as *a study of degenerate projective forms modulo the non-degenerate ones.* In this section we also describe completely the equivalence relation (excision) between degenerate projective forms which corresponds to congruence of their discriminant forms, cf. Theorem 4.8. Hence, Theorems 4.7 and 4.8 give a classification of torsion Hermitian forms.

A slightly similar approach to classification of degenerate Hermitian forms was suggested recently by E. Bayer - Fluckiger and L. Fainsilber [BF]. They showed that the classification of (degenerate) forms can be reduced to studying non-degenerate forms in the category of morphisms; note that the extended abelian category is a factor-category of the category of morphisms (we factor out the null-homotopic morphisms).

In §5 we introduce *positively and negatively definite* torsion Hermitian forms. It turns out that any non-degenerate torsion Hermitian form is a direct sum of a positively and a negatively definite forms (Theorem 5.3). It is much harder to establish the uniqueness of this decomposition; we prove the uniqueness in §7 under an additional assumption that the initial von Neumann category is *superfinite* (Theorem 7.7). The definition of superfiniteness (cf. 7.2) is analogous to the von Neumann definition of finiteness with

the only difference that here we care about the torsion objects instead of projective ones, as in the classical case. We show some examples when the superfiniteness holds; these examples include the categories which describe the growth, as in [Fa4]. Another important property of superfinite categories is given in Theorem 7.4; it states that a positively definite form is fully determined by its module.

We also study metabolic and hyperbolic torsion Hermitian forms. We show that any form is metabolic and the metabolizer is sometimes unique. But, we show, that the hyperbolicity imposes very strong requirements on the structure (the positive and the negative parts must be identical), cf. Theorem 7.10.

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Contents

§1. Dualities and Hermitian forms in categories

In this section we will briefly review a general formalism of studying Hermitian forms in additive categories, as developed in [QSS, R1, R2]. We make some adjustments and modifications, which will be convenient for the subsequent sections. In particular, we emphasize the role of the group of automorphisms of a category with duality.

In the rest of this article we will apply this general formalism to study Hermitian forms in von Neumann categories and on torsion objects of their extended abelian categories, cf. §§2 - 7.

In all these applications we will always have $\mathbf{C}$ (the field of complex numbers) as part of the automorphism group of the category; therefore we will ignore the distinction between the quadratic and the symmetric cases as in [R1, R2].

1.1. Dualities in categories. Let $\mathcal{C}$ be a category. We do not assume that $\mathcal{C}$ is abelian or additive, if not indicated explicitly.

Definition. *A duality in category $\mathcal{C}$ is a pair (D, s), consisting of a contravariant functor $D : \mathcal{C} \to \mathcal{C}$ and an isomorphism of functors $s : \mathrm{Id} \to D \circ D$, satisfying the following condition: for any object M of category $\mathcal{C}$ the morphisms $D(s_M)$ and $s_{D(M)}$ are inverse to each other:*

$$s_{D(M)} \;=\; D(s_M)^{-1}. \tag{1-1}$$

Here s_M denotes the natural isomorphism $M \to D(D(M))$ determined by the isomorphism of functors s. In particular, $s_{D(M)}$ is a morphism $D(M) \to D(D(D(M)))$. The morphism $D(s_M) : D(D(D(M))) \to D(M)$ is obtained by applying the duality functor D to s_M.

The object $D(M)$ is said to be *the dual* of M; for any morphism $f : M \to N$ in $\mathcal{C}$ we have *the dual morphism* $D(f) : D(N) \to D(M)$.

Remark. A. Ranicki [R1, R2] calls duality (D, s) *an involution* on the category $\mathcal{C}$; he also assumes (as well as in [QSS]) that there is given a canonical identification of any object M with $D(D(M))$ via s_M, so that the isomorphism of functors s becomes the identity. In this paper we prefer not doing this. In fact, we will deal here with situations when there are several different dualities in different subcategories of a given abelian category, so mentioning both D and s is essential.

The following simple observation will be useful in the sequel.

Lemma. *If $\mathcal{C}$ is an abelian with duality (D, s) so that the functor D is additive. Then D is exact, i.e. it maps any exact sequence into an exact sequence.*

Proof. First, we observe that the map $D : \hom(M, N) \to \hom(D(N), D(N))$ is bijective. If D is additive, then it maps zero morphism to zero; thus we obtain that $f : M \to N$ is a monomorphism if and only if $D(f)$ is an epimorphism. It follows that D maps kernels to cokernels and conversely. If $0 \to N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \to 0$ is an exact sequence, then β is cokernel of α and therefore $D(\beta)$ is kernel of $D(\alpha)$ which means that

$$0 \to D(N_3) \xrightarrow{D(\beta)} D(N_2) \xrightarrow{D(\alpha)} D(N_1) \to 0$$

is exact. $\square$

1.2. The symmetry group. Given a category $\mathcal{C}$ with duality (D, s), we will consider *its symmetry group* $G(\mathcal{C})$, which is defined as the set of all natural isomorphisms of the identity functor $\mathrm{Id}_{\mathcal{C}}$ to itself which are compatible with the duality (in the sense explained below). An element $\epsilon \in G(\mathcal{C})$ assigns an isomorphism $\epsilon_M : M \to M$ to any object M of $\mathcal{C}$ so that for any morphism $f : M \to N$ we have $f \circ \epsilon_M = \epsilon_N \circ f$. We say that ϵ *is compatible with duality* if for any $M \in \mathrm{Ob}(\mathcal{C})$ holds

$$s_M \circ \epsilon_M = DD(\epsilon_M) \circ s_M. \tag{1-2}$$

In other words, we require that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\epsilon_M} & M \\ {\scriptstyle s_M}\downarrow & & \downarrow{\scriptstyle s_M} \\ DD(M) & \xrightarrow[DD(\epsilon_M)]{} & DD(M). \end{array} \tag{1-3}$$

This is equivalent to

$$DD(\epsilon_M) = \epsilon_{DD(M)}. \tag{1-4}$$

The group $G(\mathcal{C})$ may clearly depend on the choice of duality (D, s) in the category $\mathcal{C}$ although we will not indicate this fact in our notation.

Remark. It is easy to see that the group $G(\mathcal{C})$ of symmetries of any category with duality is always *abelian*.

There is naturally defined *involution in* $G(\mathcal{C})$: if $\epsilon \in G(\mathcal{C})$, let us denote by $\overline{\epsilon} \in G(\mathcal{C})$ the natural transformation $\mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ given by the diagram

$$\begin{array}{ccc} M & \xrightarrow{\overline{\epsilon}_M} & M \\ {\scriptstyle s_M}\big\downarrow & & \big\downarrow{\scriptstyle s_M} \\ DD(M) & \xrightarrow[D(\epsilon_{D(M)})]{} & DD(M) \end{array} . \tag{1-5}$$

One easily checks that this diagram defines a natural transformation $\overline{\epsilon} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ and thus an element of the symmetry group $G(\mathcal{C})$. We observe the following useful formulae

$$\begin{aligned} \epsilon_{DD(M)} &= D(\overline{\epsilon}_{D(M)}), \quad \overline{\overline{\epsilon}} = \epsilon, \\ \overline{\epsilon_1 \circ \epsilon_2} \;=\; \overline{\epsilon_2} \circ \overline{\epsilon_1} \;&=\; \overline{\epsilon_1} \circ \overline{\epsilon_2} \quad \text{for any} \quad \epsilon_1, \epsilon_2 \in G(\mathcal{C}). \end{aligned} \tag{1-6}$$

1.3. Hermitian forms. Given a category $\mathcal{C}$ with duality (D, s), one studies *Hermitian forms* in $\mathcal{C}$.

Definition. *A form in a category with duality $\mathcal{C}$ is a morphism $\phi : M \to D(M)$, where M is an object of $\mathcal{C}$. ϕ is called non-degenerate if it is an isomorphism in $\mathcal{C}$. Given a form $\phi : M \to D(M)$, consider the composition*

$$\phi^\dagger \quad : M \xrightarrow{s(M)} D(D(M)) \xrightarrow{D(\phi)} D(M). \tag{1-7}$$

The form $\phi^\dagger$ is called the transpose of ϕ.

Note that

$$\phi^{\dagger\dagger} \;=\; \phi. \tag{1-8}$$

A form $\phi : M \to D(M)$ is called ϵ-*Hermitian*, where ϵ is an symmetry of the category with duality $\mathcal{C}$, i.e. $\epsilon \in G(\mathcal{C})$, if

$$\phi^\dagger \;=\; \epsilon_{D(M)} \circ \phi \;=\; \phi \circ \epsilon_M.$$

We will write this briefly as $\phi^\dagger \;=\; \epsilon\phi$. One easily checks that for $\eta \in G(\mathcal{C})$, $(\eta\phi)^\dagger = \overline{\eta}\phi^\dagger$. Hence if ϕ is ϵ-Hermitian then

$$\phi = \phi^{\dagger\dagger} = (\epsilon\phi^\dagger)^\dagger = \overline{\epsilon}\phi^\dagger = \epsilon\overline{\epsilon}\phi.$$

Conclusion. Non-degenerate ϵ-Hermitian forms exit only if the symmetry $\epsilon \in G(\mathcal{C})$ is unitary: $\epsilon\overline{\epsilon} \;=\; 1$.

Let $\phi_1 : M_1 \to D(M_1)$, and $\phi_2 : M_2 \to D(M_2)$ be two forms in category with duality $\mathcal{C}$. They are called *congruent (or isometric)* if there is an isomorphism $f : M_1 \to M_2$ such that $\phi_1 \;=\; D(f) \circ \phi_2 \circ f$.

Congruence is an equivalence relation. If two forms are congruent and one of them is ϵ-Hermitian for some $\epsilon \in G(\mathcal{C})$, then the other is also ϵ-Hermitian.

Now we will show how the problem of describing the congruence classes of ϵ-Hermitian forms may depend on ϵ. Suppose that $\eta \in G(\mathcal{C})$ and $\phi : M \to D(M)$ is an ϵ-Hermitian form for some $\epsilon \in G(\mathcal{C})$. Consider the form $\psi = \eta\phi : M \to D(M)$. Then an easy computation shows that ψ is $\epsilon\overline{\eta}\eta^{-1}$-Hermitian. It follows from the above remark that *the set of congruence classes of ϵ-Hermitian forms in $\mathcal{C}$ depends only on the class of ϵ in the factor-group*

$$\mathcal{E}(\mathcal{C}) \;=\; \{\epsilon \in G(\mathcal{C}); \epsilon\overline{\epsilon} = 1\}/1\}/\{\overline{\eta}\eta^{-1}; \eta \in G(\mathcal{C})\}. \tag{1-9}$$

We will call (1-9) *the group of types of forms* in $\mathcal{C}$.

Note that any element in $\mathcal{E}(\mathcal{C})$ is of order two: $\epsilon\overline{\epsilon} = 1$ implies that $\epsilon^2 = \overline{\eta}\eta^{-1}$ for $\eta = \overline{\epsilon}$.

1.4. Example: Duality of finitely generated projective modules. This is the most familiar example of a category with duality.

Let Λ denote a ring with involution which will be denoted $\lambda \mapsto \overline{\lambda}$. Consider the category Λ-mod of left finitely generated projective Λ-modules and Λ-homomorphisms. If M is a projective finitely generated Λ-module, then by $D(M)$ we will denote the set of all *anti-linear* homomorphisms $f : M \to \Lambda$, i.e. all additive maps satisfying

$$f(\lambda m) \;=\; f(m)\overline{\lambda} \tag{1-10}$$

for all $m \in M$ and $\lambda \in \Lambda$. A left Λ-module structure on $D(M)$ is defined as follows: for $\lambda \in \Lambda$ and $f \in D(M)$,

$$(\lambda \cdot f)(m) \;=\; \lambda f(m) \tag{1-11}$$

where $m \in M$. It is easy to see that this formula defines a structure of left Λ-module on $D(M)$. One checks also that $D(M)$ is projective and finitely generated. Any Λ-homomorphism $\phi : M \to N$ between finitely generated projective left Λ-modules induces naturally a homomorphism

$$D(\phi) : D(N) \to D(M)$$

by the rule $D(\phi)(f) = f \circ \phi$, where $f \in D(N)$. Thus we obtain a contravariant functor $D : \Lambda\text{-mod} \to \Lambda\text{-mod}$.

The canonical isomorphism

$$s_M : M \;\to\; DD(M) \tag{1-12}$$

is given by

$$m \mapsto (f \mapsto \overline{f(m)}), \quad f \in D(M). \tag{1-13}$$

One checks that, firstly, $(f \mapsto \overline{f(m)})$ is an anti-liner functional on $D(M)$, i.e. an element of $DD(M)$, and, secondly, the map (1-12) is a Λ-isomorphism. The pair (D, s) is a duality in category Λ-mod.

The group $G(\mathcal{C})$ of symmetries of this category with duality coincides with the set of all invertible elements in the center $Z(\Lambda)$ of the ring Λ, i.e. $G(\mathcal{C}) = Z(\Lambda)^*$. The involution on $G(\mathcal{C})$, defined as in subsection 1.2 (cf. (1-5)), coincides with the restriction of the original involution of Λ onto $Z(\Lambda)^*$. We obtain that the group of types of forms is

$$\mathcal{E}(\mathcal{C}) \;=\; \{\epsilon \in Z(\Lambda); \epsilon\overline{\epsilon} = 1\}/\{\overline{\eta}\eta^{-1}; \eta \in Z(\Lambda)^*\}. \tag{1-14}$$

In particular, we obtain that *the group $\mathcal{E}(\mathcal{C})$ depends only on the center $Z(\Lambda)$ and the action of the involution of Λ on the center.*

§2. Hermitian forms in von Neumann categories

In this section we will first recall the main properties of von Neumann categories, which were introduced in [GLR] and independently in [Fa3]. Our exposition will follow [Fa3], §2, §5. We show that the classification of non-degenerate Hermitian forms in a von Neumann category is immediate.

2.1. Hilbertian spaces. Recall that *a Hilbertian space* (cf. [P]) is a topological vector space H, which is isomorphic to a Hilbert space in the category of topological vector spaces. In other words, there exists a scalar product on H, such that H with this scalar product is a Hilbert space with the originally given topology. Such scalar products are called *admissible*. Given one admissible scalar product $\langle\ ,\ \rangle$ on H, any other admissible scalar product is given by

$$\langle x, y\rangle_1 \;=\; \langle Ax, y\rangle, \quad x, y \in H, \tag{2-1}$$

where $A : H \to H$ is an invertible positive operator $A^* = A, \quad A > 0$. Hilbertian spaces naturally appear as Sobolev spaces of sections of vector bundles, cf. [P].

Let us denote by $\mathfrak{Hilb}$ the category of Hilbertian spaces and continuous linear maps. This category has an obvious duality (D, s), where $D(H)$ is defined as the space H^* of all anti-linear continuous functionals on H (i.e the set of all continuous $\mathbf{R}$-linear maps $\phi : H \to \mathbf{C}$, such that $\phi(\lambda h) = \overline{\lambda}\phi(h)$ for all $\lambda \in \mathbf{C}$ and $h \in H$; here the bar denotes the complex conjugation). We consider the action of $\mathbf{C}$ on $D(H) = H^*$ given by $(\lambda \cdot \phi)(h) = \phi(\overline{\lambda}^* \cdot h)$ for all $h \in H$. The canonical isomorphism

$$s_H : H \to DD(H) = H^{**} \tag{2-2}$$

is $h \mapsto (\phi \mapsto \overline{\phi(h)})$, where $h \in H$, and $\phi \in H^*$. One checks that the condition (1-1) is satisfied and so we have a category with duality.

2.2. Hilbertian von Neumann categories. *A Hilbertian von Neumann category* is an additive subcategory $\mathcal{C}$ of $\mathfrak{Hilb}$ with the following properties:

(1) *for any $H \in \mathrm{Ob}(\mathcal{C})$ the dual space H^* is also an object of $\mathcal{C}$ and there is a $\mathcal{C}$-isomorphism $\phi : H \to H^*$ such that the formula*

$$\langle x, y\rangle = \phi(x)(y), \quad x, y \in H$$

defines an admissible scalar product on H;

(2) *for any $H \in \mathrm{Ob}(\mathcal{C})$ the isomorphism (2-2) also belongs to $\mathcal{C}$;*

(3) *the adjoint of any morphism in $\mathcal{C}$ also belongs to $\mathcal{C}$;*

(4) *the kernel* $\ker f = \{x \in H; f(x) = 0\}$ *of any morphism $f : H \to H'$ in $\mathcal{C}$ and the natural inclusion* $\ker f \to H$ *belong to $\mathcal{C}$;*

(5) *for any $H, H' \in \mathrm{Ob}(\mathcal{C})$ the set of morphisms*

$$\hom_{\mathcal{C}}(H, H') \subset \hom_{\mathfrak{Hilb}}(H, H') = \mathcal{L}(H, H')$$

is a linear subspace closed with respect to the weak topology.

Hence, objects of $\mathcal{C}$ have structure of Hilbertian spaces and possibly some additional structure, and morphisms of $\mathcal{C}$ are (faithfully) represented by bounded linear maps.

Condition (5) is similar to the well-known condition in the definition of von Neumann algebras, which explains our term. Recall, that the weak topology on the space of bounded linear operators $f : H \to H'$ is given by the family of seminorms

$$p_{\phi,x}(f) = |\langle \phi, f(x)\rangle|, \quad \text{where} \quad \phi \in H', \quad x \in H. \tag{2-3}$$

Note also, that given any object $H \in \mathrm{Ob}(\mathcal{C})$, a choice of $\mathcal{C}$-admissible scalar product on H, determines an involution on the algebra $\hom_{\mathcal{C}}(H, H)$ (adjoint operator) and the space $\hom_{\mathcal{C}}(H, H)$ considered with this involution is a von Neumann algebra.

The above conditions (1) - (5) imply the following properties:

(6) *The closure of the image* $\mathrm{cl}(\mathrm{im}\, f)$ *of any morphism* $f : H \to H'$ *in* $\mathcal{C}$ *and also the natural projection* $H' \to H'/\,\mathrm{cl}(\mathrm{im}\, f)$ *belong to* $\mathcal{C}$.

(7) *Suppose that* $H' \subset H$ *is a closed subspace. If* H', H *and the inclusion* $H' \to H$ *belong to* $\mathcal{C}$ *then the orthogonal complement* ${H'}^{\perp}$ *with respect to an admissible scalar product on* H *and the inclusion* ${H'}^{\perp} \to H$ *belong to* $\mathcal{C}$.

Remark. The duality structure in $\mathfrak{Hilb}$, described in 2.1, induces a duality structure in any von Neumann category $\mathcal{C} \subset \mathfrak{Hilb}$. This follows from conditions (1), (2) and (3) of subsection 2.2. Hence *any Hilbertian von Neumann category is a category with duality.*

2.3. Example. The simplest example of a Hilbertian von Neumann category is the following. Let $\mathcal{A}$ be an algebra over $\mathbf{C}$ with involution, which on $\mathbf{C}$ coincides with the complex conjugation. *A Hilbertian representation of* $\mathcal{A}$ is a Hilbertian topological vector space H supplied with a left action of $\mathcal{A}$ by continuous linear maps $\mathcal{A} \to \mathcal{L}(H, H)$. A morphism $f : H \to H'$ between two Hilbertian representations of $\mathcal{A}$ is defined as a bounded linear map commuting with the action of the algebra $\mathcal{A}$. There is a canonical duality in the category of all Hilbertian representations of a given $*$-algebra $\mathcal{A}$. Namely, given a Hilbertian representation H, consider the space $D(H) = H^*$ of all anti-linear continuous functionals on H. Consider the following action of $\mathcal{A}$ on $D(H) = H^*$: if $\phi \in H^*$ and $\lambda \in \mathcal{A}$ then $(\lambda \cdot \phi)(h) = \phi(\lambda^* \cdot h)$ for all $h \in H$. The canonical isomorphism

$$s_H : H \to DD(H) = H^{**} \tag{2-4}$$

is given by $h \mapsto (\phi \mapsto \overline{\phi(h)})$, where $h \in H$, and $\phi \in H^*$. Category of all Hilbertian representations of $\mathcal{A}$ is a von Neumann category.

2.4. Finite von Neumann categories. An object H of a Hilbertian von Neumann category $\mathcal{C}$ will be called *finite* if the only closed $\mathcal{C}$-submodule $H_1 \subset H$ which is isomorphic to H in $\mathcal{C}$ is $H_1 = H$.

A Hilbertian von Neumann category will be called *finite* iff all its objects are finite.

We will use the following property of finite categories.

2.5. Proposition. *Let* $\mathcal{C}$ *be a finite von Neumann category and let* $\alpha : H \to H'$ *be an injective morphism of* $\mathcal{C}$ *with dense image. Then* H *is isomorphic to* H' *in* $\mathcal{C}$. *For any injective nonzero morphism* $\beta : H'' \to H'$ *the preimage* $\beta^{-1}(\alpha(H)) \subset H''$ *is dense (and so non-empty).*

Proof. The first statement was proven in [Fa1], Lemma 2.3. By Lemma 2.4 of [Fa1] the preimage $\beta^{-1}(\alpha(H)) \subset H''$ is not empty. If it is not dense, consider a complement

$H''' \subset H''$ to the closure of $\beta^{-1}(\alpha(H))$. Let $\beta' : H''' \to H'$ be the restriction of β. Then the preimage ${\beta'}^{-1}(\alpha(H))$ is empty, in contradiction with the established above property. $\square$

2.6. More examples of von Neumann categories. .

Example 1. The following is one of the most important examples of von Neumann categories. Let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space H. Denote by $\mathcal{A}$ be the commutant of $\mathcal{B}$. We will consider the following category $\mathcal{C}$ of Hilbertian representations of $\mathcal{A}$. Objects of $\mathcal{C}$ are in one-to-one correspondence with projections $e \in M(n) \otimes \mathcal{B}$, $e^2 = e$, for some n, where $M(n)$ is the $n \times n$-matrix algebra. For each projection e the corresponding Hilbert representation of $\mathcal{A}$ is $e(\mathbf{C}^n \otimes H)$. If e_1 and e_2 are two projections, then the set of morphisms $\hom_{\mathcal{C}}(e_1(\mathbf{C}^{n_1} \otimes H), e_2(\mathbf{C}^{n_2} \otimes H))$ is the set of all bounded linear maps of the form $e_2 b e_1 : e_1(\mathbf{C}^{n_1} \otimes H) \to e_2(\mathbf{C}^{n_2} \otimes H)$ where b is given by an $n_1 \times n_2$-matrix with entries in $\mathcal{B}$. CC is a von Neumann category.

Example 2. Let Z be a locally compact Hausdorff space and let μ be a positive Radon measure on Z. Let $\mathcal{A}$ denote the algebra $L^\infty_{\mathbf{C}}(Z, \mu)$ (the space of essentially bounded μ-measurable complex valued functions on Z, in which two functions equal locally almost everywhere, are identical). The involution on $\mathcal{A}$ is given by the complex conjugation. We will construct a category $\mathcal{C}$ of Hilbert representations of $\mathcal{A}$ as follows. The objects of $\mathcal{C}$ are in one-to-one correspondence with the μ-measurable fields of finite-dimensional Hilbert spaces $\xi \to \mathcal{H}(\xi)$ over (Z, μ), cf. [Di], part II, chapter 1. For any such measurable field of Hilbert spaces, the corresponding Hilbert space is the direct integral

$$H = \int^{\oplus} \mathcal{H}(\xi) d\mu(\xi) \tag{2-5}$$

defined as in [Di], part II, chapter 1. The algebra $\mathcal{A}$ acts on the Hilbert space (2-5) by pointwise multiplication.

Suppose that we have two μ-measurable finite-dimensional fields of Hilbert spaces $\xi \to \mathcal{H}(\xi)$ and $\xi \to \mathcal{H}'(\xi)$ over Z. Then we have two corresponding Hilbert spaces, H and H', given as direct integrals (2-5). We define the set of morphisms $\hom_{\mathcal{C}}(H, H')$ as the set of all bounded linear maps $H \to H'$ given by *decomposable linear maps*

$$T = \int^{\oplus} T(\xi) d\mu(\xi), \tag{2-6}$$

where $T(\xi)$ is *an essentially bounded measurable field of linear maps* $T(\xi) : \mathcal{H}(\xi) \to \mathcal{H}'(\xi)$, cf. [Di], part II, chapter 2.

The kernel of any decomposable linear map as above can be represented as the direct integral of a finite-dimensional field of Hilbert spaces and so condition (c) of section 2.4 is satisfied. Condition (a) of section 2.4 is also satisfied since the adjoint of the map T given by (2-6) is

$$T^* = \int^{\oplus} T(\xi)^* d\mu(\xi) \tag{2-7}$$

(by [Di], part II, chapter 2, §3, Proposition 3). Condition (d) of section 2.4 is satisfied as follows from Theorem 1 of [Di], part II, chapter 2, §5.

Thus, we obtain a von Neumann category. This category is finite. Note that this category $\mathcal{C}$ depends only on the *class of the measure* μ.

Other examples of von Neumann categories can be found in [Fa3], §2.

2.7. Group of types of forms. Since any von Neumann category $\mathcal{C}$ is a category with duality, we may consider its symmetry group $G(\mathcal{C})$, cf. 1.2. The corresponding group of types of forms $\mathcal{E}(\mathcal{C})$ (cf. (1-9)) is trivial in all examples mentioned above. For instance, in example 1 of section 2.6 the symmetry group $G(\mathcal{C})$ coincides with the group $Z(\mathcal{A})^*$ of invertible elements in the center of the von Neumann algebra $\mathcal{A}$. If $\mathcal{A}$ is a factor, then $Z(\mathcal{A}) = \mathbf{C}$ and $\mathcal{E}(\mathcal{C}) = 0$. If $\mathcal{A}$ is not a factor, then $Z(\mathcal{A})$ can be represented as the ring $L^\infty_{\mathbf{C}}(Z, \mu)$ of essentially bounded measurable functions on a measure space (Z, μ). Then any unitary element $\epsilon \in Z(\mathcal{A})$ is represented by a function with values in the circle S^1. Given such ϵ, it is clear that one can find another function η on Z with values in S^1 such that $\eta^2 = \epsilon$. Thus $\mathcal{E}(\mathcal{C}) = 0$.

2.8. Hermitian forms in von Neumann categories. Let $\mathcal{C}$ be a finite von Neumann category. It is a category with duality (cf. Remark in 2.2) and so we may apply to $\mathcal{C}$ the formalism developed in §1.

Any Hermitian form $\phi : H \to H^*$ in $\mathcal{C}$ determines a *continuous scalar product*

$$\langle\ ,\ \rangle_\phi : H \times H \to \mathbf{C},$$

given by the formula

$$\langle h, h'\rangle_\phi \;=\; \phi(h)(h'), \quad \text{for} \quad h, h' \in H,$$

which satisfies the following properties:

$$\begin{aligned}
\langle h, h'\rangle_\phi &= \overline{\langle h', h\rangle}_\phi \qquad \text{(because } \phi \text{ is Hermitian)},\\
\langle \lambda h, h'\rangle_\phi &= \langle h, \lambda^* h'\rangle_\phi, \quad \lambda \in \mathcal{A}, \quad \text{(since } \phi \text{ commutes with } \mathcal{A}),\\
\langle \lambda h, h'\rangle_\phi &= \lambda\langle h, h'\rangle_\phi \quad \text{for} \quad \lambda \in \mathbf{C}.\\
\langle h, h'\rangle_\phi &\le C \cdot ||h|| \cdot ||h'||, \quad \text{(since } \phi \text{ is continuous)}.
\end{aligned} \tag{2-8}$$

Here we assume that we have chosen a $\mathcal{C}$-admissible scalar product $\langle\ ,\ \rangle$ on H (i.e. an admissible scalar product which appears in 2.2.(1)) and that $||\cdot||$ denotes the corresponding norm.

Note that if the kernel $\phi^{-1}(0)$ is zero, then the image $\phi(H) \subset H^*$ is dense, as follows from finiteness of H.

If ϕ is Hermitian then $\langle h, h\rangle_\phi$ is real. We will say that a form ϕ as above is *positively definite* if $\langle h, h\rangle_\phi$ is a positive real number for all $h \in H,\ h \neq 0$. Similarly, we will say that a form ϕ is *negatively definite* if $-\phi$ is positively definite.

We will call a Hermitian form $\phi : H \to H$ *weakly invertible* is it is injective $\phi^{-1}(0) = 0$.

2.9. Proposition. *In a finite von Neumann category $\mathcal{C}$, any weakly non-degenerate Hermitian form $\phi : H \to H^*$ can be represented as an orthogonal sum*

$$\phi \;=\; \phi_+ \oplus \phi_-, \quad H = H_+ \oplus H_- \tag{2-9}$$

of a positively definite form $\phi_+ : H_+ \to H_+^*$ and a negatively definite form $\phi_- : H_- \to H_-^*$.

Proof. Fix a $\mathcal{C}$-admissible scalar product $\langle\ ,\ \rangle$ on H. Then the form ϕ determines a bounded self-adjoint injective operator $A_\phi \in \hom_{\mathcal{C}}(H, H)$ as above. By the spectral theorem we have the following decomposition

$$A_\phi \;=\; \int_{-\infty}^{\infty} \lambda dE_\lambda,$$

where E_λ are the spectral projectors determined by A_ϕ. Note that all these projectors E_λ belong to $\hom_{\mathcal{C}}(H,H)$; in particular, they commute with the action of $\mathcal{A}$. Consider the mutually orthogonal projections $P_- = E_0$ and $P_+ = E_N - E_0$, where N is sufficiently large, and the corresponding orthogonal decomposition

$$H \ = \ H_+ \oplus H_-, \quad \text{where} \quad H_\pm = P_\pm H. \tag{2-10}$$

The restriction of the form ϕ on $H_\pm$ is positively (negatively) definite. □

2.10. Proposition. *Suppose that a Hermitian form $\phi : H \to H^*$ in a finite von Neumann category $\mathcal{C}$ is non-degenerate (i.e. ϕ is an isomorphism in $\mathcal{C}$). Then the decomposition (2-9) is unique. This means that for any other representation of H as the direct sum of two closed subspaces $H = H'_+ \oplus H'_-$, such that $H'_\pm \in \mathrm{Ob}(\mathcal{C})$, and the form ϕ is positively definite on H'_+ and negatively definite on H'_-, and H'_+ is orthogonal to H'_- with respect to ϕ, then H_+ is isomorphic to H'_+ and H_- is isomorphic to H'_- in $\mathcal{C}$; moreover, the restriction $\phi|_{H_\pm}$ is congruent to $\phi|_{H'_\pm}$.*

Proof. Let L be an arbitrary closed subspace of H such that the form ϕ is positively definite on L. Each $x \in L$ can be uniquely represented as $x = x_+ + x_-$ with $x_\pm \in H_\pm$. Consider the projection $\pi : L \to H_+$, where $x \mapsto x_+$. We claim that this map $\pi : L \to H_+$ is an injection with closed image. In fact, if there exists a sequence (x_n) with $x_n \in L$, $||x_n|| = 1$, and $\pi(x_n) \to 0$, then we obtain

$$0 < \ \langle x_n, x_n\rangle_\phi \le \langle \pi(x_n), \pi(x_n)\rangle_\phi \to 0,$$

and therefore $\langle x_n, x_n\rangle_\phi \to 0$, which contradicts the assumption that ϕ is non-degenerate.

Thus, if we have two decompositions $H = H_+ \oplus H_-$ and $H = H'_+ \oplus H'_-$, then H_+ can be mapped into H'_+ and H'_+ can be mapped into H_+. From finiteness of $\mathcal{C}$ it now follows, that H_+ is isomorphic to H'_+ in $\mathcal{C}$.

To show that $\phi|_{H_\pm}$ is congruent to $\phi|_{H'_\pm}$ we may use an easy fact that *any two non-degenerate positively definite forms on a given object of $\mathcal{C}$ are congruent.* □

§3. Duality for torsion Hilbertian modules

In this section we construct a duality in the abelian category of torsion Hilbertian modules.

3.1. Abelian extension of a von Neumann category. Given a Hilbertian von Neumann category $\mathcal{C}$, there is a bigger category $\mathcal{E}(\mathcal{C})$, containing $\mathcal{C}$ as a full subcategory [Fa], [Fa1]. The advantage of $\mathcal{E}(\mathcal{C})$ is that it is *an abelian category.* A brief description of the construction is given below; we refer to [Fa3] for more details.

An object of the category $\mathcal{E}(\mathcal{C})$ is defined as a morphism $(\alpha : A' \to A)$ in the original category $\mathcal{C}$. Given a pair of objects $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ of $\mathcal{E}(\mathcal{C})$, a *morphism* $\mathcal{X} \to \mathcal{Y}$ in category $\mathcal{E}(\mathcal{C})$ is an equivalence class of morphisms $f : A \to B$ of category $\mathcal{C}$ such that $f \circ \alpha = \beta \circ g$ for some morphism $g : A' \to B'$ in $\mathcal{C}$. Two morphisms $f : A \to B$ and $f' : A \to B$ of $\mathcal{C}$ represent *identical morphisms $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{E}(\mathcal{C})$* iff $f - f' = \beta \circ F$ for some morphism $F : A \to B'$ of category $\mathcal{C}$. The morphism $\mathcal{X} \to \mathcal{Y}$, represented by $f : A \to B$, is denoted by

$$[f] : (\alpha : A' \to A) \ \to \ (\beta : B' \to B) \quad \text{or by} \quad [f] : \mathcal{X} \to \mathcal{Y}. \tag{3-1}$$

Composition of morphisms in $\mathcal{E}(\mathcal{C})$ is defined as composition of the corresponding morphisms f in the category $\mathcal{C}$.

The category $\mathcal{E}(\mathcal{C})$ is an abelian category, cf. [Fa3], Proposition 1.7. It is called *the abelian extension of the category* $\mathcal{C}$.

It is shown in [Fa3], section 1.4, that any object $\mathcal{X}$ of $\mathcal{E}(\mathcal{C})$ is isomorphic in $\mathcal{E}(\mathcal{C})$ to an object $(\alpha : A' \to A)$, where the morphism α is injective.

3.2. Torsion Hilbertian modules. An object $\mathcal{X} = (\alpha : A' \to A)$ of the extended category $\mathcal{E}(\mathcal{C})$ is called *torsion* iff the image of α is dense in A.

We will denote by $\mathcal{T}(\mathcal{C})$ the full subcategory of $\mathcal{E}(\mathcal{C})$ generated by all torsion objects. $\mathcal{T}(\mathcal{C})$ is called *the torsion subcategory of* $\mathcal{E}(\mathcal{C})$. It is shown in [Fa3], §3, that:

if $\mathcal{C}$ *is a finite von Neumann category, then* $\mathcal{T}(\mathcal{C})$ *is an abelian category.*

Objects of $\mathcal{T}(\mathcal{C})$ are called *torsion Hilbertian modules.*

3.3. Duality for torsion Hilbertian modules. Our aim now is to construct duality in the category $\mathcal{T}(\mathcal{C})$ of torsion Hilbertian modules, i.e. a contravariant functor

$$\mathfrak{e} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C}) \tag{3-2}$$

and an isomorphism of functors

$$s : \mathrm{Id} \to \mathfrak{e} \circ \mathfrak{e}, \tag{3-3}$$

cf. 1.1. For any torsion Hilbertian module $\mathcal{X} = (\alpha : A' \to A)$ (where α is injective) define the *dual module* $\mathfrak{e}(\mathcal{X})$ by

$$\mathfrak{e}(\mathcal{X}) \;=\; (D(\alpha) : D(A) \to D(A')), \tag{3-4}$$

where D denotes the duality functor in $\mathcal{C}$, cf. sections 2.1, 2.2.

Suppose now that we have two torsion objects $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ with injective α and β and let $[f] : \mathcal{X} \to \mathcal{Y}$ be a morphism represented by a diagram

$$\begin{array}{ccc} (A' & \xrightarrow{\alpha} & A) \\ & & \big\downarrow f \\ (B' & \xrightarrow{\beta} & B). \end{array} \tag{3-5}$$

According to definition §3.1, there exists a morphism $h : A' \to B'$ making this diagram commutative; this h is in fact unique, because of injectivity of β. We define the *dual morphism of* $[f]$ as the morphism

$$\mathfrak{e}([f]) \;=\; [D(h)] : \mathfrak{e}(\mathcal{Y}) \;\to \mathfrak{e}(\mathcal{X}). \tag{3-6}$$

It is represented by the diagram

$$\begin{array}{ccc} (D(B) & \xrightarrow{D(\beta)} & D(B')) \\ & & \big\downarrow D(h) \\ (D(A) & \xrightarrow[D(\alpha)]{} & D(A')). \end{array} \tag{3-7}$$

Since different morphisms f may represent the same morphism in $\mathcal{T}(\mathcal{C})$, cf. §3.1, we have to check correctness of the above definition. If $F : A \to B'$ is an arbitrary morphism, then the morphism $f' = f + \beta \circ F$ represents the same morphism $[f] = [f']$ in $\mathcal{E}(\mathcal{C})$. Then the corresponding morphism h' is $h' = h + F \circ \alpha$ and thus

$$D(h') \;=\; D(h) \;+\; D(\alpha) \circ D(F),$$

which means that $D(h)$ and $D(h')$ represent the same morphism in $\mathcal{T}(\mathcal{C})$, cf. [Fa3].

The isomorphism of functors $s : \mathrm{Id} \to \mathfrak{e} \circ \mathfrak{e}$ is given as follows. For any torsion Hilbertian module $\mathcal{X} = (\alpha : A' \to A)$ define

$$s_{\mathcal{X}} : \mathcal{X} \to \mathfrak{e}(\mathfrak{e}(\mathcal{X})) \quad \text{by} \quad s_{\mathcal{X}} = [s_A], \tag{3-8}$$

where s_A is the canonical isomorphism $A \to DD(A)$ in the category $\mathcal{C}$), cf. sections 2.1, 2.2. We have the following commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle s_{A'}}\downarrow & & \downarrow{\scriptstyle s_A} \\ DD(A') & \xrightarrow[DD(\alpha)]{} & DD(A), \end{array} \tag{3-9}$$

representing the morphism $s_{\mathcal{X}} : \mathcal{X} \to \mathfrak{e}(\mathfrak{e}(\mathcal{X}))$.

§4. Classification of Hermitian forms on torsion Hilbertian modules

This section plays a central role in the paper; here we give a classification of non-degenerate Hermitian forms on torsion Hilbertian modules.

We refer to the paper of E. Bayer - Fluckiger and L. Fainsilber [BF] where a problem of classification of degenerate Hermitian forms was studied. Their results are similar in the spirit to the results of this section.

4.1. Let $\mathcal{C}$ be a finite von Neumann category and let $\mathcal{T}(\mathcal{C})$ be the torsion subcategory of the extended abelian category $\mathcal{E}(\mathcal{C})$, cf. 3.2. We will study Hermitian forms in $\mathcal{T}(\mathcal{C})$, considered with the duality described in §3.

Let $\mathcal{X} = (\alpha : A' \to A)$ be a torsion object of $\mathcal{E}(\mathcal{A})$ represented by an *injective* morphism α with dense image. Then a Hermitian form on $\mathcal{X}$ is given by a morphism

$$\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X}), \quad \text{with} \quad \phi^\dagger = \phi \tag{4-1}$$

(cf. §1). According to the definitions of §3 and [Fa3], the morphism ϕ is represented by a commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ D(A) & \xrightarrow[D(\alpha)]{} & D(A') \end{array} \tag{4-2}$$

in $\mathcal{C}$. Clearly, f determines h uniquely (since α is injective and has dense image and hence $D(\alpha)$ is also injective). The condition $\phi^\dagger = \phi$ means (translating the general definitions of §1) that

$$f - h^\dagger \;=\; D(\alpha) \circ F, \tag{4-3}$$

where $F : A \to D(A)$ is a morphism of $\mathcal{C}$ and $h^\dagger$ denotes the composition

$$A \xrightarrow{s_A} DD(A) \xrightarrow{D(h)} D(A'),$$

i.e. the transpose of h in $\mathcal{C}$, cf. (1-7). Note that (4-3) implies that F is anti-symmetric, i.e. $F^\dagger = -F$.

Definition. Pair of morphisms (f, h) as in diagram (4-2) above, will be called *symmetric presentation of the Hermitian form* $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ if

$$f \;=\; h^\dagger. \tag{4-4}$$

4.2. Lemma. *Any Hermitian form in $\mathcal{T}(\mathcal{C})$ admits a symmetric presentation (not unique).*

Proof. Let a pair (f, h) be an arbitrary presentation of a morphism ϕ as above. Denote

$$f_1 \;=\; f - \frac{1}{2}D(\alpha) \circ F \quad \text{and} \quad h_1 \;=\; h - \frac{1}{2}F \circ \alpha,$$

where F satisfies (4-3). Then the obtained pair of morphisms (f_1, h_1) also represents ϕ and we have $f_1 \;=\; h_1^\dagger$. $\square$

4.3. A symmetric presentation (f, h) of a form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ can be also viewed as follows. The map $f : A \to D(A')$ defines a pairing $\langle\ ,\ \rangle : A \times A' \to \mathbf{C}$ (where $\langle a, a' \rangle \;=\; f(a)(a')$ for $a \in A, a' \in A'$) which is continuous as a function of two variables, $\mathbf{C}$-linear with respect to the first variable and skew-linear with respect to the second variable, and satisfies $\langle \lambda a, a' \rangle \;=\; \langle a, \lambda^* a' \rangle$ for any $a \in A$, $a' \in A'$ and $\lambda \in \mathcal{A}$. Besides, symmetricity implies that the induced form $\{\ ,\ \} : A' \times A' \to \mathbf{C}$, where $\{x, y\} \;=\; \langle \alpha(x), y \rangle$ for $x, y \in A'$, is *Hermitian*, i.e. $\{x, y\} \;=\overline{\{y, x\}}$.

4.4. Discriminant forms. The easiest way of constructing Hermitian forms on torsion objects in $\mathcal{T}(\mathcal{C})$ consists in the following. Suppose that we are given a Hilbertian module $A \in \mathrm{Ob}(\mathcal{C})$ and a Hermitian form in category $\mathcal{C}$ (cf. §2) $\alpha : A \to D(A)$. We will suppose that α is injective, but not necessarily surjective. Then α represents a torsion object

$$\mathcal{X} = \;(\alpha : A \to D(A)) \in \mathrm{Ob}(\mathcal{T}(\mathcal{C})) \tag{4-5}$$

and, moreover, we have the Hermitian form

$$\phi_\alpha : \mathcal{X} \to \mathfrak{e}(\mathcal{X}), \tag{4-6}$$

given by the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & D(A) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[\alpha^\dagger]{} & D(A). \end{array} \tag{4-7}$$

The form ϕ_α is clearly non-degenerate.

Thus, *any Hermitian form* α *in category* $\mathcal{C}$ *produces a non-degenerate Hermitian form on a torsion object in* $\mathcal{T}(\mathcal{C})$. We will call the obtained form ϕ_α the *discriminant form of* α, because of similarity of this construction with the well-known construction of discriminant forms in number theory, cf. [N].

It is clear that the discriminant form ϕ_α of any non-degenerate form α is trivial (i.e. it has $\mathcal{X} = 0$). Roughly, *the discriminant form* ϕ_α *measures the "way of degeneration" of the form* α.

The construction of discriminant form can be made even simpler if we suppose that a $\mathcal{C}$-admissible scalar product on a Hilbertian module A is specified. In this situation we may use this scalar product to identify the dual module $D(A)$ with A and so the Hermitian form α is now represented by a self-adjoint operator

$$\alpha : A \to A, \qquad \alpha^* \ = \ \alpha, \qquad \alpha \in \hom_{\mathcal{C}}(A, A). \tag{4-8}$$

Here we have $\mathcal{X} \ = \ (\alpha : A \to A)$ and the discriminant form $\phi_\alpha : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ will be represented by the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ A & \xrightarrow[\alpha]{} & A. \end{array} \tag{4-9}$$

4.5. Example. Consider the von Neumann algebra $\mathcal{A} = L^\infty(S^1)$ acting on $L^2(S^1)$ by multiplication. Let $\mathcal{C}$ denote the von Neumann category, which is obtained by applying the construction of Example 2 in section 2.6 to the circle S^1 with the Lebesgue measure μ.

The space $L^2(S^1)$ has a canonical $\mathcal{C}$-admissible scalar product. Therefore, any real valued function $\alpha \in L^\infty(S^1)$, which does not vanish a subset of positive measure, determines (by multiplication) an operator $\alpha : L^2(S^1) \to L^2(S^1)$, which gives (as explained in (4-9)) a Hermitian form $\phi_\alpha : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in category $\mathcal{T}(\mathcal{C})$, where $\mathcal{X} \ = \ (\alpha : L^2(S^1) \to L^2(S^1))$.

Note that the same form $\phi_\alpha \ = \ \phi_\beta$ in $\mathcal{T}(\mathcal{C})$ will be described by any other function $\beta \in L^\infty(S^1)$ of the form

$$\beta(z) \ = \ \alpha(z) \ + \ \alpha^2(z)F(z), \quad z \in S^1, \tag{4-10}$$

for arbitrary real valued $F \in L^\infty(S^1)$.

Now we face the main question: what do the functions $\alpha(z)$ and $\beta(z)$ have in common? In fact, choosing $F(z)$ we may make $\beta(z)$ to be arbitrary *far from the zeros of* $\alpha(z)$.

To understand the situation, suppose for simplicity, that the real valued function α on S^1 is continuous and vanishes at a single point $z_0 \in S^1$. Then it is clear that the equation $\gamma^2(z) \ = \ 1 \ + \ \alpha(z)F(z)$ can be solved *in a neighborhood* U of z_0. Thus we obtain that $\beta(z) \ = \ \alpha(z)\gamma(z)^2$ for $z \in U$ and so *the forms in* $\mathcal{C}$ *represented by the functions* α *and* β *become isometric, if we ignore the points of the circle outside* U.

This shows that despite the drastic global changes which can be performed on the function α and which do not influence the corresponding Hermitian form ϕ_α in $\mathcal{T}(\mathcal{C})$, the behavior of the function $\alpha(z)$ *near the zeros* $\alpha(z) = 0$ remains essentially the same. We will see in the next subsection, that a similar (and more precise!) statement can be made in a completely general situation.

4.6. Excision. We will describe now a procedure of excision for Hermitian forms in $\mathcal{T}(\mathcal{C})$, which is similar to the procedure of excision on objects of the extended category, cf. [Fa3], 1.4.

Suppose that the given Hermitian form $\alpha : A \to D(A)$ can be represented as an orthogonal sum $A = P \oplus Q, \quad P \perp Q$, where P and Q are orthogonal with respect to the form α, and the restriction $\alpha|_P$ is non-degenerate, i.e. $\alpha|_P : P \to D(P)$ is an isomorphism. Then the Hermitian form

$$\beta = \alpha|_Q : Q \to D(Q)$$

produces discriminant form ϕ_β on $(\beta : Q \to D(Q))$, which is obviously isometric to ϕ_α in $\mathcal{T}(\mathcal{C})$. We will say that *the form β is obtained from α by excision with respect to $P \subset A$.*

As an example of excision, which we will use frequently, consider the following construction. Suppose that we have a $\mathcal{C}$-admissible scalar product on A and a self-adjoint operator $\alpha \in \hom_{\mathcal{C}}(A, A)$. It defines a Hermitian form in $\mathcal{T}(\mathcal{C})$, cf. 4.4. Let

$$\alpha \; = \; \int_{-\infty}^{\infty} \lambda dE_\lambda \tag{4-11}$$

be the spectral representation of α. Given $\epsilon > 0$, consider the decomposition $A = P \oplus Q$, where

$$P \; = \; E_{-\epsilon}A \; + \; (1 - E_\epsilon)A, \quad Q \; = \; (E_\epsilon - E_{-\epsilon})A. \tag{4-12}$$

Then α maps P into P and induces an isomorphism $P \to P$. Also, P and Q are orthogonal with respect to α. Thus we can make an excision with respect to P to get a form $\alpha|_Q : Q \to Q$.

Note that the norm of $\alpha|_Q$ is less or equal than ϵ.

The following two theorems will be our main technical results allowing to reduce the questions about non-degenerate torsion forms in $\mathcal{T}(\mathcal{C})$ to questions about (degenerate!) forms in the initial von Neumann category *ca*.

4.7. Theorem. *Any non-degenerate Hermitian form in $\mathcal{T}(\mathcal{C})$ is isometric to a discriminant form of a Hermitian form in $\mathcal{C}$.*

4.8. Theorem. *Suppose that two Hermitian forms $\alpha : A \to D(A)$ and $\beta : B \to D(B)$ in category $\mathcal{C}$ (with injective α and β) produce isometric discriminant forms ϕ_α and ϕ_β in $\mathcal{T}(\mathcal{C})$. Then the forms α and β admit excisions, which are isometric as forms in $\mathcal{C}$.*

4.9. Proof of Theorem 4.7. Without loss of generality, we may assume that we are given a non-degenerate Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ represented by a commutative diagram

$$\begin{array}{ccc} (\alpha : A & \longrightarrow & A) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (\alpha : A & \longrightarrow & A) \end{array} \tag{4-13}$$

(a symmetric presentation) where we suppose that a $\mathcal{C}$-admissible scalar product on A has been chosen and α is injective, self-adjoint $\alpha^* = \alpha$, and has dense image.

Suppose that f in diagram (4-13) is an isomorphism. Then we have a discriminant form

$$\begin{array}{ccc} (f\alpha : A & \longrightarrow & A) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ (\alpha f^* : A & \longrightarrow & A), \end{array} \tag{4-14}$$

which is congruent to (4-13) because of the diagram

$$\begin{array}{ccc} (\alpha : A & \longrightarrow & A) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (f\alpha : A & \longrightarrow & A) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ (\alpha f^* : A & \longrightarrow & A) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ (\alpha : A & \longrightarrow & A). \end{array} \tag{4-15}$$

Thus, to prove the Theorem, we have to show that, given diagram (4-13), representing an isomorphism in $\mathcal{T}(\mathcal{C})$, we can "replace" it by a diagram with f isomorphic. The "replacement" will mean certain "excision".

As **the first step**, we want to show that we may make f injective. Let $B \subset A$ denote $\overline{\mathrm{im}(f^*)}$ and $B' \subset A$ denote $\overline{\mathrm{im}(f)}$; we consider B and B' with the induced from A scalar products. Denote by $i : B \to A$ and $i' : B' \to A$ the inclusions, and $\pi : A \to B$ and $\pi' : A \to B'$ will denote the orthogonal projections.

We will also need the following map $j' : B' \to A$. To define it, note, that the orthogonal complement to B' coincides with $\ker(f^*)$ and the orthogonal complement to B is $\ker(f)$. Since diagram (4-13) represents a monomorphism, using Proposition 1.6 of [Fa3], we find that α maps $\ker(f^*)$ *onto* $\ker(f)$. Therefore, given $x \in B'$ we may write $\alpha(x) = a + b$, where $a \in B$ and $b \in \ker(f)$. Now, as explained above, $b = \alpha(b')$ for some $b' \in \ker(f^*)$ and we set

$$j'(x) = x - b'. \tag{4-16}$$

The map j' is clearly a monomorphism.

Consider the torsion Hilbertian module

$$\mathcal{Y} = (\beta : B' \to B), \quad \text{where} \quad \beta = \pi \circ \alpha \circ i'.$$

We have the following morphism in $\mathcal{T}(\mathcal{C})$

$$\begin{array}{ccc} (\beta : B' & \longrightarrow & B) \\ {\scriptstyle j'}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ (\alpha : A & \longrightarrow & A). \end{array} \tag{4-17}$$

This implies that β is injective and has dense image. Using Proposition 1.6 of [Fa3] we check that the kernel and the cokernel of the morphism of $\mathcal{E}(\mathcal{C})$, represented by (4-17) are trivial and so (4-17) represents an isomorphism in $\mathcal{E}(\mathcal{C})$.

The initially given form ϕ on $\mathcal{X}$ induces a form $\psi : \mathcal{Y} \to \mathfrak{e}(\mathcal{Y})$ via isomorphism (4-17). We find that ψ is given by the diagram

$$\begin{array}{ccc} (\beta : B' & \longrightarrow & B) \\ {\scriptstyle g^*}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ (\beta^* : B & \longrightarrow & B'), \end{array} \tag{4-18}$$

where $g = j'^* \circ f \circ i : B \to B'$. We only have to check now that g is *injective*. But g clearly coincides with the action of f as the map $\overline{\mathrm{im}(f^*)} \to \overline{\mathrm{im}(f)}$. The last map is obviously injective on $\mathrm{im}(f^*)$; now the result follows because of finiteness assumptions, using [Fa3], Proposition 2.4. Thus, we may replace the initial form (4-13) by (4-18) where g is injective.

As **the second step**, we will assume that f in (4-13) is injective and will make it isomorphism by performing further excisions.

Since diagram (4-13) represents an isomorphism, using Proposition 1.6 of [Fa3], we find that the sequence

$$0 \to A \xrightarrow{\alpha \oplus -f^*} A \oplus A \xrightarrow{(f,\alpha)} A \to 0 \tag{4-19}$$

is exact in $\mathcal{E}(\mathcal{C})$. Therefore it splits, since it consists of projective objects. Thus, there exist morphisms $\sigma : A \to A$ and $\delta : A \to A$, such that

$$\mathrm{id}_A \;=\; \sigma\alpha \;+\; \delta f^*. \tag{4-20}$$

This is the crucial point of the proof; intuitively, (4-20) means that f^* *and* α *cannot be small at the same places.* Choose $\epsilon > 0$ such that $\epsilon \cdot ||\sigma|| < 1$. Using decomposition (4-12), we may construct a splitting

$$A = \; P \oplus \; Q, \quad P \perp Q, \tag{4-21}$$

such that $\alpha(P) = P$, $\alpha(Q) \subset Q$ and the norm of the restriction of α on Q is less than ϵ, i.e. $||\alpha|_Q|| < \epsilon$. Denoting by $i_Q : Q \to A$ and by $\pi_Q : A \to Q$ the corresponding inclusion and projection, we will have (using (4-20)) that the morphism $\pi_Q \delta f^* i_Q : Q \to Q$ is an isomorphism. Therefore, $f^* i_Q : Q \to A$ is injective with closed image.

Let's show that the (injective) map $f i_Q : Q \to A$ also has closed image. Since (4-13) represents an epimorphism in $\mathcal{E}(\mathcal{C})$, we know that $\overline{\mathrm{im}(\alpha) + \mathrm{im}(f)} = A$ (cf. Proposition 1.6 of [Fa3]). We have $A = \overline{f(P)} \oplus \overline{f(Q)}$ and thus we obtain (since we have already proved that $f^*(Q) \subset A$ is closed) that $\alpha(f^*(Q)) + f(Q) = \overline{f(Q)}$. Thus, since $\alpha(f^*(Q)) \subset f(Q)$, we get $f(Q) = \overline{f(Q)}$.

We now want to show that the map $\pi_Q : Q' \to Q$ is an isomorphism; it is clearly equivalent to the statement that $\pi_Q f^* i_Q : Q \to Q$ is an isomorphism. We have

$$\pi_Q f i_P = \alpha \pi_Q f^* \alpha^{-1} i_P$$

(because $f\alpha = \alpha f^*$ and α preserves P and Q and is an isomorphism $\alpha : P \to P$). Therefore,

$$||\pi_Q f i_P|| \leq C\epsilon$$

for some constant $C > 0$ (since $||\alpha|_Q|| < \epsilon$), and thus

$$||\pi_P f^*(q)|| \leq C\epsilon \cdot ||q|| \quad \text{for} \quad q \in Q.$$

On the other hand, since we know that $f^* : Q \to A$ is injective with closed image, holds

$$||f^*(q)|| \geq C' \cdot ||q||, \quad \text{for} \quad q \in Q$$

for some constant $C' > 0$. Hence we obtain

$$\begin{aligned} ||\pi_Q f^*(q)|| \geq ||f^*(q)|| - ||\pi_P f^*(q)|| \geq \\ \geq (C' - C\epsilon) \cdot ||q||, \quad \text{for} \quad q \in Q. \end{aligned} \tag{4-22}$$

Note that the constants C and C' may be assumed independent of ϵ and of the decomposition $A = P + Q$ and so for small enough $\epsilon > 0$ we will have $\pi_q f^* : Q \to Q$ isomorphism because of (4-22).

Now we may finish the proof as follows. Denote $Q' = F(Q)$. The diagram

$$\begin{array}{ccc} Q & \xrightarrow{f\alpha i_Q} & Q' \\ {\scriptstyle i_Q}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ A & \xrightarrow{\alpha} & A \end{array} \tag{4-23}$$

represents an isomorphism in $\mathcal{E}(\mathcal{C})$ (apply Proposition 1.6 of [Fa3]), and the induced Hermitian form on $(f\alpha i_Q : Q \to Q')$ is obviously given by the diagram

$$\begin{array}{ccc} Q & \xrightarrow{f\alpha i_Q} & Q' \\ {\scriptstyle \pi_{Q'} i_Q}\downarrow & & \downarrow{\scriptstyle \pi_Q i_{Q'}} \\ Q' & \xrightarrow{(f\alpha i_Q)^*} & Q \end{array} \tag{4-24}$$

We have shown above that $\pi_Q i_{Q'} : Q' \to Q$ is an isomorphism. The result now follows from the argument in the very beginning of the proof.

4.10. Proof of Theorem 4.8. Suppose that some $\mathcal{C}$-admissible scalar products on A and on B are chosen so that we may consider $\alpha : A \to A$ and $\beta : B \to B$ as self-adjoint operators, injective, with dense images. An isometry between the discriminant forms ϕ_α and ϕ_β is then represented by commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\beta]{} & B. \end{array} \tag{4-25}$$

The fact, that it is an isometry means that it is an isomorphism in $\mathcal{E}(\mathcal{C})$ and also the composition morphism given by the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\beta]{} & B \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ A & \xrightarrow{\alpha} & A, \end{array} \tag{4-26}$$

equals ϕ_α (defined as (4-9)). Therefore, there exists a morphism $F : A \to A$ of $\mathcal{C}$ such that

$$g^* \circ f = \mathrm{id}_A + \alpha \circ F, \quad f^* \circ g = \mathrm{id}_A + F \circ \alpha. \tag{4-27}$$

Now, choose $\epsilon > 0$ such that $\epsilon \cdot ||F|| < 1$, and perform excision (4-12) on $(\alpha : A \to A)$ with respect to this ϵ. As the result we will obtain $(\alpha|_Q : Q \to Q)$ and (4-25) produces the isometry

$$\begin{array}{ccc} Q & \xrightarrow{\alpha|_Q} & Q \\ {\scriptstyle g|_Q}\downarrow & & \downarrow{\scriptstyle f|_Q} \\ B & \xrightarrow[\beta]{} & B. \end{array} \tag{4-28}$$

From the second equation in (4-27) we see that $g|_Q$ is injective with closed image (this is similar to the argument used in the proof of Theorem 4.7).

To simplify our notations, we will now assume that the initial isometry (4-25) has the property

$$||\alpha|| \cdot ||F|| < 1 - \delta \quad \text{for some small} \quad \delta > 0, \tag{4-29}$$

where F satisfies (4-27). Then, as we have already mentioned, g is injective with closed image. We will denote $B' = g(B) \subset A$.

We will finish the proof of the Theorem by showing:

(i) *The Hermitian form α is congruent to the form β' induced by β on $B' \subset B$;*

(ii) *The form induced by β on ${B'}^{\perp}$, the orthogonal complement to B' with respect to β, is non-degenerate.*

Indeed, (i) would imply that β' is *weakly non-degenerate*, i.e. the morphism $\beta' : B' \to D(B')$ is injective; therefore, $A = B' \oplus {B'}^{\perp}$, and so using (ii) we obtain that α *is isometric to an excision of β.*

On the other hand, once (i) is obtained, (ii) would follow from the fact that (4-25) is an isomorphism in $\mathcal{E}(\mathcal{C})$. So, we only need to prove (i).

Consider the form A induced by g from β, i.e. $g^*\beta g$. From (4-26) and (4-27) we obtain

$$g^*\beta g = \alpha f^* g = \alpha + \alpha F \alpha. \tag{4-30}$$

Our aim (i) will be achieved if we will show that the right hand side of (4-30) can be represented in the form

$$\alpha \ + \ \alpha F \alpha \ = \ h^* \alpha h, \quad \text{where} \quad h : A \to A \tag{4-31}$$

is an $\mathcal{C}$-isomorphism.

Let us represent α in the form $\alpha = s\gamma^2$, where the morphisms $s, \gamma : A \to A$ satisfy

$$s^2 = 1, \quad s^* = s, \quad \gamma^* = \gamma, \quad \gamma > 0, \quad \gamma s = s\gamma.$$

The morphism γ is the positive square root of $|\alpha|$, and s is the *sign* of α. The operators γ and s are obtained by

$$s = \int_{-\infty}^{\infty} \operatorname{sign}(\lambda) dE_\lambda, \quad \gamma = \int_{-\infty}^{\infty} |\lambda| dE_\lambda,$$

where E_λ are the spectral projectors of the spectral decomposition of α. Note, that all the spectral projectors and the operators s and γ, given by the above integrals, are morphisms of $\mathcal{C}$ since $\hom_{\mathcal{C}}(A, A)$ is a von Neumann algebra, and so we may use the general results of [Di].

Then we have $\alpha + \alpha F \alpha = s\gamma(1 + \gamma F s \gamma)\gamma$.

Now, let us find $\mathcal{C}$-isomorphisms $h, h_1, h_2 : A \to A$, such that

$$\begin{aligned} 1 + \gamma F s \gamma &= h_1^2, \\ 1 + \gamma^2 F s &= h_2^2, \\ 1 + F s \gamma^2 &= h^2. \end{aligned} \tag{4-32}$$

We will show that we may construct h, h_1 and h_2 such that the following identities will hold

$$s h_2 = h^* s, \tag{4-33}$$

$$\gamma h_1 = h_2 \gamma, \tag{4-34}$$

$$\gamma h = h_1 \gamma. \tag{4-35}$$

To construct δ we observe that the spectrum of $1+\gamma F s\gamma$ is contained (because of (4-29)) inside the circle $\Gamma_\delta = \{\lambda; |\lambda - 1| = 1 - \delta\}$. Thus we may use the functional calculus to define

$$h_1 = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{\lambda^{1/2}}{\lambda - (1 + \gamma F s \gamma)} d\lambda, \tag{4-36}$$

where $\lambda^{1/2}$ denotes the branch of the square root which is obtained by cutting along the negative real axis; we will use the fact that it commutes with the complex conjugation. We define the operators h_2 and h by the similar integrals

$$h_2 = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{\lambda^{1/2}}{\lambda - (1 + \gamma^2 F s)} d\lambda \tag{4-37}$$

and

$$h = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{\lambda^{1/2}}{\lambda - (1 + F s \gamma^2)} d\lambda. \tag{4-38}$$

One easily checks that the formulae (4-33), (4-34), (4-35) are clearly satisfied. For example to prove (4-34) one observes that

$$\gamma \cdot (\lambda - (1 + \gamma F s \gamma))^{-1} = (\lambda - (1 + \gamma^2 F s))^{-1} \gamma.$$

Now we have using (4-33), (4-34), (4-35)

$$\begin{aligned} \alpha \ + \ \alpha F\alpha \ &= s\gamma h_1^2\gamma \\ &= s(\gamma h_1)(h_1\gamma) \\ &= s(h_2\gamma)(\gamma h) \\ &= h^* s\gamma^2 h \\ &= h^*\alpha h. \end{aligned} \tag{4-39}$$

The fact that h is an isomorphism follows by comparing the equation $1 + F\alpha = h^2$ (cf. (4-33)) with inequality (4-29).

This completes the proof. □

§5. Positively and negatively definite torsion Hermitian forms

5.1. Definition. *A non-degenerate Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ will be called positively (negatively) definite if it can be represented as the discriminant form ϕ_α of a positively (negatively) defined Hermitian form $\alpha : A \to D(A)$* in $\mathcal{C}$.

Equivalently, a torsion Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is positively definite, if and only if any Hermitian form $\alpha : A \to D(A)$ in $\mathcal{C}$ which produces ϕ as the discriminant form $\phi = \phi_\alpha$, can be represented as an orthogonal sum $\alpha = \alpha_1 \perp \alpha_2$, where α_1 is non-degenerate and α_2 is positively definite; this follows from Theorem 4.8.

5.2. Lemma. *Any non-degenerate torsion Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$, which at the same time is positively and negatively definite, is trivial: $\mathcal{X} = 0$.*

Proof. Suppose that a non-degenerate torsion Hermitian form can be represented as the discriminant form of a positively definite form $\alpha : A \to D(A)$ and of a negatively definite form $\beta : B \to D(B)$. Then by Theorem 4.8, α and β have isometric excisions. Since any excision of a positively (negatively) defined form is positively (negatively, correspondingly) defined, and since a positively definite form can be isometric to a negatively definite form if and only if the both forms are trivial, we obtain that the forms α and β admit trivial excisions. Hence the maps $\alpha : A \to D(A)$ and $\beta : B \to D(B)$ are isomorphisms and so $\mathcal{X} = 0$. □

5.3. Theorem. *Any non-degenerate torsion Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ can be represented as an orthogonal sum*

$$\phi \ = \ \phi_+ \ \perp \ \phi_-, \tag{4-41}$$

where $\phi_+ : \mathcal{X}_+ \to \mathfrak{e}(\mathcal{X}_+)$ is a positively definite and $\phi_- : \mathcal{X}_- \to \mathfrak{e}(\mathcal{X}_-)$ is a negatively definite torsion Hermitian forms.

Proof. It follows directly from Theorem 4.7. Namely, given a non-degenerate torsion Hermitian form $\phi : \mathcal{X} \to D(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$, by Theorem 4.7 it can be represented as the discriminant form ϕ_α of a Hermitian form $\alpha \in \hom_{\mathcal{C}}(A, D(A))$. If we chose a $\mathcal{C}$-admissible scalar product on A, then α can be identified with a self-adjoint operator $\alpha : A \to A$. Using the spectral theorem, we can represent α as $\alpha_+ \oplus \alpha_-$ where $\alpha_\pm : A_\pm \to A_\pm$ is a positive (negative) operator $\alpha_\pm \in \hom_{\mathcal{C}}(A, A)$. Then $\phi \ = \ \phi_+ \oplus \phi_-$, where $\phi_\pm$ is the discriminant form of $\alpha_\pm$. □

We will prove later uniqueness of the decomposition of Theorem 5.3 (cf. Theorems 7.7 and 7.4), but we will impose some additionally assumption on the category (superfiniteness).

§6. Metabolic and hyperbolic forms

In this section we will assume that $\mathcal{C}$ is a finite von Neumann category, and thus $\mathcal{T}(\mathcal{C})$ is an abelian category, cf. [Fa3].

Recall a general terminology concerning Hermitian forms. A form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ is called *metabolic* if there is an inclusion $i : \mathcal{Y} \to \mathcal{X}$ (in the sense of the category theory) such that $\mathcal{Y}$ coincides with its annihilator $\mathcal{Y}^{\perp}$, i.e. with the kernel of

$$\mathcal{X} \xrightarrow{\phi} \mathfrak{e}(\mathcal{X}) \xrightarrow{\mathfrak{e}(i)} \mathfrak{e}(\mathcal{Y}). \tag{6-1}$$

Such subobject $\mathcal{Y} \subset \mathcal{X}$ with $\mathcal{Y} = \mathcal{Y}^{\perp}$ is called *a metabolizer.*

A form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is called *hyperbolic* if it has a metabolizer which is a direct summand.

We will see that torsion Hermitian forms in $\mathcal{T}(\mathcal{C})$ are all metabolic, and, moreover, in many cases the metabolizer is unique. We will also see that they are rarely hyperbolic.

6.1. Proposition. *Any non-degenerate torsion Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ is metabolic.*

Proof. Because of Theorem 5.3, it is enough to prove metabolicity of positively defined torsion forms. By Theorem 4.7 and observations made in 4.4, we may assume that the given form ϕ is represented by diagram (4-9), where $\alpha : A \to A$ is a self-adjoint positive operator, $\alpha \in \hom_{\mathcal{C}}(A, A)$. We may find a positive square root of α, i.e. $\beta \in \hom_{\mathcal{C}}(A, A)$ with $\beta^2 = \alpha$, $\beta^* = \beta$, $\beta > 0$. Denote $\mathcal{Y} = (\beta : A \to A)$, and let $i : \mathcal{Y} \to \mathcal{X}$ be given by the diagram

$$\begin{array}{ccc} (A & \xrightarrow{\beta} & A) \\ \downarrow{\scriptstyle 1} & & {\scriptstyle \beta}\downarrow \\ (A & \xrightarrow{\alpha} & A). \end{array}$$

Then the composition (6-1) equals

$$\begin{array}{ccc} (A & \xrightarrow{\alpha} & A) \\ \downarrow{\scriptstyle \beta} & & {\scriptstyle 1}\downarrow \\ (A & \xrightarrow{\beta} & A) \end{array}$$

and using Proposition 1.6 of [Fa], one easily checks that its kernel coincides with $i : \mathcal{Y} \to \mathcal{X}$. $\square$

6.2. Proposition. *Let $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ be a non-degenerate Hermitian form, which is positively (or negatively) definite. Then in $\mathcal{X}$ there is a unique metabolizer.*

Proof. Suppose that $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is presented as the discriminant form of a positively definite Hermitian form $\alpha : A \to A$. As in the proof of 6.1 we suppose that A has

a specified admissible scalar product and $\alpha \in \hom_{\mathcal{C}}(A,A)$ is self-adjoint and positive. Then any inclusion $i : \mathcal{Y} \to \mathcal{X}$ in $\mathcal{T}(\mathcal{C})$ can be represented by a diagram

$$\begin{array}{ccc} (A & \xrightarrow{\beta} & A) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \gamma} \\ (A & \xrightarrow{\alpha} & A), \end{array}$$

where $\alpha = \gamma\beta$. Using Proposition 1.6 of [Fa] we observe that $\mathcal{Y} \subset \mathcal{Y}^\perp$ if and only if γ can be represented as $\gamma = \beta^*\delta$ for some $\delta \in \hom_{\mathcal{C}}(A,A)$. Thus we have $\alpha = \beta^*\delta\beta$. Now, $\alpha^* = \alpha$ implies $\delta^* = \delta$. Also, δ is clearly positive. The inclusion $\mathcal{Y}^\perp \to \mathcal{X}$ is represented by

$$\begin{array}{ccc} (A & \xrightarrow{\gamma} & A) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \beta} \\ (A & \xrightarrow{\alpha} & A) \end{array}$$

and the inclusion $\mathcal{Y} \to \mathcal{Y}^\perp$ is represented by

$$\begin{array}{ccc} (A & \xrightarrow{\beta} & A) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \delta} \\ (A & \xrightarrow{\gamma} & A). \end{array}$$

Again, using Proposition 1.6 of [Fa] we find that $\mathcal{Y} = \mathcal{Y}^\perp$ (i.e. $\mathcal{Y}$ is a metabolizer) if and only if δ is an isomorphism in $\mathcal{C}$. Now we may write $\delta = \delta_1^2$ with $\delta_1^* = \delta_1$ and $\delta_1 > 0$. Therefore we see that our metabolizer $\mathcal{Y}$ coincides with the one constructed in Proposition 6.1 (which corresponds to $\delta_1\beta$ taken instead of β). □

§7. Superfinite von Neumann categories

In this section we introduce an additional finiteness assumption on the initial von Neumann category $\mathcal{C}$, which we call *superfiniteness.* We classify completely torsion Hermitian forms in $\mathcal{T}(\mathcal{C})$, assuming that $\mathcal{C}$ is superfinite. More precisely, we prove (cf. Theorems 7.4 and 7.7) that any such form determines canonically two torsion modules $\mathcal{X}_+$ and $\mathcal{X}_-$ (the positive and the negative parts) and, conversely, the congruence class of the form is determined by the isomorphisms types of $\mathcal{X}_+$ and $\mathcal{X}_-$.

7.1. Proposition. *For a finite von Neumann category $\mathcal{C}$ the following properties are equivalent:*

(i) *any epimorphism $\mathcal{X} \to \mathcal{X}$, where $\mathcal{X} \in \mathrm{Ob}(\mathcal{T}(\mathcal{C}))$, is an isomorphism;*
(ii) *any monomorphism $\mathcal{X} \to \mathcal{X}$, where $\mathcal{X} \in \mathrm{Ob}(\mathcal{T}(\mathcal{C}))$, is an isomorphism;*
(iii) *in any commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\alpha} & A, \end{array} \qquad (7\text{-}1)$$

in $\mathcal{C}$, where α is injective and f is an isomorphism in $\mathcal{C}$, the morphism g is also an isomorphism in $\mathcal{C}$.

Proof. Equivalence of (i) and (ii) easily follows from existence of the duality $\mathfrak{e}$ in $\mathcal{T}(\mathcal{C})$, cf. 3.3.

Suppose that (i) and (ii) are satisfied. Then any diagram (7-1) can be viewed as a morphism $\phi : \mathcal{X} \to \mathcal{X}$ of the torsion subcategory $\mathcal{T}(\mathcal{C})$ where $\mathcal{X} = (\alpha : A \to A)$. If f is an isomorphism then ϕ is an epimorphism, and thus it is an isomorphism by (i). This implies that g is an isomorphism in $\mathcal{C}$ (as easy follows from Proposition 1.6 of [Fa3]).

Conversely, suppose that we know that (iii) is always satisfied. Using the arguments used in the proof of Theorem 4.7, we obtain that any epimorphism $\mathcal{X} \to \mathcal{X}$ can be represented by a diagram of form (7-1), where $\mathcal{X} = (\alpha : A \to A)$, α is injective and f is an isomorphism. Then (iii) implies that g is an isomorphism as well, and thus the given epimorphism $\mathcal{X} \to \mathcal{X}$ is an isomorphism. Therefore, (iii) implies (i). $\square$

7.2. Definition. A finite von Neumann category $\mathcal{C}$ will be called *superfinite* if the equivalent conditions of Proposition 7.1 hold for $\mathcal{C}$.

The diagram (7-1) can be viewed as follows. We may think of α as defining a new scalar product on A and then the operators f and g describe "the same" operator, which is bounded with respect to two different norms. Such operators are studied in [GK], for example, where they called bibounded. I am grateful to V. Matsaev, who explained this to me. Our condition of superfiniteness (7-1) is equivalent to the statement that the spectra of f and g coincide.

As example in [GK], chapter 5, §6 show, the category of Hilbert spaces and bounded linear maps is not superfinite.

7.3. Example. Consider the von Neumann category $\mathcal{C}$ described as Example 2 in section 2.6. A diagram of type (7-1) in this category looks as follows.

We are given a locally compact Hausdorff space Z with a positive Radon measure μ, and the Hilbert space A is the direct integral (2-5) of a measurable field of finite dimensional Hilbert spaces $\xi \to \mathcal{H}(\xi)$, where $\xi \in Z$, cf. [Di], part II. We will assume that $\mathcal{H}(\xi)$ is nonzero for any $\xi \in Z$. The morphisms α, f, and g of (7-1) are given by decomposable linear maps (2-6) constructed out of essentially bounded measurable fields of linear maps $\mathcal{H}(\xi) \to \mathcal{H}(\xi)$, cf. 2.7. We will denote the corresponding measurable fields of linear maps by

$$T_\alpha(\xi), \quad T_f(\xi), \quad T_g(\xi) : \ \mathcal{H}(\xi) \to \mathcal{H}(\xi), \qquad \xi \in Z. \tag{7-2}$$

Since α is supposed to be injective, we know that $T_\alpha(\xi)$ is an isomorphism for almost all $\xi \in Z$. Since we have the identity

$$T_g(\xi) = T_\alpha(\xi)^{-1} \circ T_f(\xi) \circ T_\alpha(\xi) \tag{7-3}$$

for almost all ξ, we obtain that $T_f(\xi)$ and $T_g(\xi)$ have the same spectrum for almost all ξ. We assumed that f is an isomorphism, so the field $\xi \mapsto T_f(\xi)^{-1}$ is essentially bounded and therefore the spectrum of the operator $T_g(\xi)$ does not approach zero. More precisely, we obtain that there exists $r > 0$ such that the spectrum of $T_g(\xi)$ does not intersect the circle $\{\lambda; |\lambda| \leq r\}$ for almost all $\xi \in Z$. Also, the norm of $T_g(\xi)$ has a uniform upper bound (since g is bounded). Thus we obtain that

$$||T_g(\xi)^{-1}|| \leq ||T_g(\xi)||^{-1} \leq 1/r$$

for almost all $\xi \in Z$ and therefore g is an isomorphism in category $\mathcal{C}$.

This shows that the von Neumann category of this example is superfinite.

One may conjecture that *any finite von Neumann category is superfinite.*

Here is the first result, whose proof use superfiniteness.

7.4. Theorem. *If $\mathcal{C}$ is a superfinite von Neumann category then any two non-degenerate positively definite torsion Hermitian forms $\phi : \mathcal{X}_1 \to \mathfrak{e}(\mathcal{X}_1)$ and $\psi : \mathcal{X}_2 \to \mathfrak{e}(\mathcal{X}_2)$ are congruent if and only if the underlying objects $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic as objects of $\mathcal{T}(\mathcal{C})$.*

Proof. It is clear that congruence of ϕ and ψ implies isomorphism between $\mathcal{X}_1$ and $\mathcal{X}_2$ and we only need to prove the inverse statement.

Without loss of generality, using Theorems 4.7 and 4.8, we may assume that the forms ϕ and ψ are given as follows. Let A be an object of $\mathcal{C}$ with a particular choice of a $\mathcal{C}$-admissible scalar product $\langle\ ,\ \rangle$ on A. Let $\alpha_\phi, \alpha_\psi : A \to A$ be the self-adjoint positive operators, $\alpha_\phi, \alpha_\psi \in \hom_{\mathcal{C}}(A, A)$ such that the forms ϕ and ψ are the discriminant forms corresponding to α_ϕ and α_ψ correspondingly as explained in subsection 4.4. Then we have $\mathcal{X}_1 = (\alpha_\phi : A \to A)$ and $\mathcal{X}_2 = (\alpha_\psi : A \to A)$, and since we know that $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic, we obtain (using Proposition 3.6 of [Fa3]) that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_\phi} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\alpha_\psi} & A, \end{array} \tag{7-4}$$

where f and g are isomorphisms in $\mathcal{C}$. Now we have $g\alpha_\phi = \alpha_\psi f$ and therefore $f^*\alpha_\psi = \alpha_\phi g^*$ and $f^* g\alpha_\phi = \alpha_\phi g^* f$. To be able to apply Lemma 7.5 below to conclude that the spectrum of f^*g is contained in an interval of the positive real axis of the form (ϵ, N) for some small $\epsilon > 0$ and for some large $N > 0$. We only have to note that $\alpha_\phi \geq 0$ and $(f^*g)\alpha_\phi = f^*\alpha_\psi f \geq 0$ using the fact that ϕ and ψ are positively definite.

Define

$$k = \frac{1}{2\pi i}\int_\Gamma \frac{\sqrt{\lambda}}{\lambda - f^*g}d\lambda, \tag{7-5}$$

where Γ is the boundary of the rectangle with vertices $\epsilon - i\delta$, $N - i\delta$, $N + i\delta$ and $\epsilon + i\delta$ for some $\delta > 0$. Here we use the branch of the $\sqrt{\lambda}$, obtained by cutting along the negative real axis, *which commutes with the complex conjugation.* Then, using the functional calculus [DS], chapter VII, §3, we have

$$k^2 = f^*g \tag{7-6}$$

and, in particular, k is a $\mathcal{C}$-isomorphism; moreover,

$$k\alpha_\phi = \alpha_\phi k^*. \tag{7-7}$$

To prove the last formula observe that

$$k^* = \frac{1}{2\pi i}\int_\Gamma \frac{\sqrt{\lambda}}{\lambda - g^*f}d\lambda \tag{7-8}$$

and now (7-7) follows from the identity

$$(\lambda - f^*g)^{-1}\alpha_\phi = \alpha_\phi(\lambda - g^*f)^{-1}.$$

Thus we obtain

$$f^*\alpha_\psi f = f^*g\alpha_\phi = k^2\alpha_\phi = k\alpha_\phi k^*, \tag{7-9}$$

which shows that ϕ and ψ are congruent.

Therefore the torsion forms ϕ and ψ are congruent. This completes the proof. $\square$

Here is the Lemma which we used in the above proof.

7.5. Lemma. *Suppose that $\mathcal{C}$ is a superfinite von Neumann category and α and β are operators acting in an object $A \in \mathrm{Ob}(\mathcal{C})$, where $\alpha, \beta \in \hom_{\mathcal{C}}(A, A)$. Suppose that we have specified an admissible scalar product $\langle\ ,\ \rangle$ on A such that β and $\beta\alpha$ are self-adjoint and positive. Then the spectrum of α is contained in the non-negative part of the real axis.*

Proof. I am very thankful to V. Matsaev, who explained to me that this statement *is not true* in the category of Hilbert spaces: he showed to me examples, constructed in [GK], chapter 5, §6, which do not satisfy Lemma 7.5. Here is the only place, where we will use the assumption of superfiniteness of our category $\mathcal{C}$.

Denote by $\langle\ ,\ \rangle_\beta$ the scalar product on A determined by β, where $\langle h, h'\rangle_\beta = \langle\beta(h), h'\rangle$, (for $h, h' \in A$) and by $||\cdot||_\beta$ the corresponding norm on A. We have for $h \in A$, $h \neq 0$,

$$\langle\alpha(h), h\rangle_\beta\ >,\ 0 \tag{7-10}$$

i.e. α is self-adjoint and positive with respect to β-scalar product. If β is invertible, then the Lemma follows from the well-known fact that the positive operator in Hilbert space has non-negative spectrum.

Now we use Lemma 7.6 below to conclude that there exists a bounded linear operator $f : A \to A$ with $f \in \hom_{\mathcal{C}}(A, A)$ such that the following digram

$$\begin{tikzcd} A \arrow[r, "\beta^{1/2}"] \arrow[d, "\alpha"'] & A \arrow[r, "\beta^{1/2}"] \arrow[d, "f"'] & A \arrow[d, "\alpha^*"] \\ A \arrow[r, "\beta^{1/2}"] & A \arrow[r, "\beta^{1/2}"] & A \end{tikzcd} \tag{7-11}$$

is commutative. From this commutative diagram we see that f is self-adjoint. Thus the spectrum of f belongs to the positive real axis. Therefore, for any $\lambda \in \mathbf{C}$, which is not real and positive, $f - \lambda$ is invertible. Therefore, we may apply the definition of superfiniteness (and Proposition 7.1) to the diagram

$$\begin{tikzcd} A \arrow[r, "\beta^{1/2}"] \arrow[d, "\alpha-\lambda"'] & A \arrow[d, "f-\lambda"] \\ A \arrow[r, "\beta^{1/2}"] & A, \end{tikzcd}$$

to conclude that $\alpha - \lambda$ is invertible. Therefore, we obtain that the spectrum of α is contained in the positive real axis. $\square$

7.6. Lemma. *Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha^2"] \arrow[d, "f"'] & A \arrow[d, "g"] \\ A \arrow[r, "\alpha^2"] & A \end{tikzcd}$$

in a finite von Neumann category, where A is supplied with an admissible scalar product and α is self-adjoint and positive. Then this diagram can be completed to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A \arrow[r, "\alpha"] \arrow[d, "h"'] & A \arrow[d, "g"] \\ A \arrow[r, "\alpha"] & A \arrow[r, "\alpha"] & A, \end{tikzcd}$$

where $h \in \hom_{\mathcal{C}}(A, A)$.

Note that h in the above diagram is clearly unique since α has dense image (as follows from finiteness of $\mathcal{C}$).

Proof. Existence of a bounded linear map h such the above diagram is commutative, is equivalent to Lemma 1.1 of [DLS]. (Note that this Lemma of [DLS] is a generalization of a well-known theorem of M.G. Krein, cf. also [L], Theorem I, or [D]).

In fact, in order to interpret the above mentioned results of [DLS] in this form, we note that we may identify the inclusion of A into its completion with respect to the α-norm $||x||_\alpha = ||\alpha(x)||$ with the morphism $\alpha : A \to A$ and after this identification the cited result can be used immediately.

We claim now that the constructed operator h must belong to the von Neumann algebra $\hom_{\mathcal{C}}(A, A)$. To show this, denote by η_μ, where $\mu > 0$, the following function of a real variable $\lambda > 0$

$$\eta_\mu(\lambda) = \begin{cases} 1, & \text{for} \quad \lambda < \mu \\ \lambda, & \text{for} \quad \lambda \geq \mu. \end{cases} \tag{7-12}$$

Note that the function $\lambda \mapsto \eta_\mu(\lambda)^{-1}$ is bounded on the spectrum of α. Now we see that h is the weak limit

$$h = \lim_{\mu \to 0} \ \alpha \circ f \circ (\int_0^\infty \eta_\mu(\lambda)^{-1} dE_\lambda) \tag{7-13}$$

of the operators belonging to $\hom_{\mathcal{C}}(A, A)$, where E_λ is the spectral projection from the spectral decomposition of α. $\square$

The following statement is the main result of this section. Together with Theorem 5.3 it gives a complete classification of torsion Hermitian forms.

7.7. Theorem. *Let $\mathcal{C}$ be a superfinite von Neumann category. Let $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ and $\psi : \mathcal{Y} \to \mathfrak{e}(\mathcal{Y})$ be two torsion Hermitian forms in $\mathcal{T}(\mathcal{C})$. Let $\mathcal{X} = \mathcal{X}_+ \perp \mathcal{X}_-$ and $\mathcal{Y} = \mathcal{Y}_+ \perp \mathcal{Y}_-$ be the decompositions into the positively and negatively definite parts, which exist as asserted by Theorem 5.3. Then the initial forms ϕ and ψ are congruent if and only if $\mathcal{X}_+$ is isomorphic to $\mathcal{Y}_+$ and $\mathcal{X}_-$ is isomorphic to $\mathcal{Y}_-$ as objects of $\mathcal{T}(\mathcal{C})$.*

We will first prove the following Lemma:

7.8. Lemma. *Suppose that $\mathcal{C}$ is a superfinite von Neumann category. If $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is a non-degenerate positive torsion Hermitian form and $\psi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is a positive torsion Hermitian form then $\phi + \psi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is non-degenerate and positive.*

Proof. We defined in §4 the notion of positivity only for non-degenerate Hermitian forms. The form ψ above in not supposed to be non-degenerate and its positivity means the following. If $\mathcal{K}$ denotes the kernel of $\psi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ then ψ naturally defines a non-degenerate form on $\mathcal{X}/\mathcal{K}$. We will say that ψ is positive iff this non-degenerate form on $\mathcal{X}/\mathcal{K}$ is positive.

Suppose that ϕ is given as the discriminant form of $\alpha : A \to A$, cf. Theorem 4.7. Here A is an object of $\mathcal{C}$ with a choice of an admissible scalar product and $\alpha \in \hom_{\mathcal{C}}(A, A)$ is self-adjoint and positive. The morphism ϕ is therefore represented by the diagram

$$\begin{array}{ccc} (A & \xrightarrow{\alpha} & A) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ (A & \xrightarrow{\alpha} & A). \end{array}$$

The other form $\psi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is represented by a diagram

$$\begin{array}{ccc} (A & \xrightarrow{\alpha} & A) \\ {\scriptstyle \beta^*}\downarrow & & \downarrow{\scriptstyle \beta} \\ (A & \xrightarrow{\alpha} & A) \end{array}$$

(the symmetric representation), cf. Lemma 4.2. Here $\alpha\beta^* = \beta\alpha$ and so $\beta\alpha$ is self-adjoint. Now, because of positivity of ψ we may choose β so that $\beta\alpha$ is non-negative. The form $\phi + \psi$ is represented by the diagram

$$\begin{array}{ccc} (A & \xrightarrow{\alpha} & A) \\ {\scriptstyle 1+\beta^*}\downarrow & & \downarrow{\scriptstyle 1+\beta} \\ (A & \xrightarrow{\alpha} & A) \end{array}$$

and the Lemma will be proven if we will show that $1 + \beta$ is an isomorphism in $\mathcal{C}$.

By Lemma 7.6 there exists $g \in \hom_{\mathcal{C}}(A, A)$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha^{1/2}} & A & \xrightarrow{\alpha^{1/2}} & A \\ {\scriptstyle \beta^*}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha^{1/2}} & A & \xrightarrow{\alpha^{1/2}} & A \end{array}$$

commutes. We have $g\alpha^{1/2} = \alpha^{1/2}\beta^*$ and $\beta\alpha^{1/2} = \alpha^{1/2}g$ which imply that g is self-adjoint. Since $\beta\alpha = \alpha^{1/2}g\alpha^{1/2} > 0$, we have $g > 0$. Therefore $1 + g : A \to A$ is an isomorphism in $\mathcal{C}$. Now considering the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha^{1/2}} & A & \xrightarrow{\alpha^{1/2}} & A \\ {\scriptstyle 1+\beta^*}\downarrow & & {\scriptstyle 1+g}\downarrow & & \downarrow{\scriptstyle 1+\beta} \\ A & \xrightarrow{\alpha^{1/2}} & A & \xrightarrow{\alpha^{1/2}} & A \end{array}$$

and using superfiniteness (cf. Proposition 7.1) we obtain that β^*+1 and $\beta+1$ are both isomorphisms. $\square$

7.9. Proof of Theorem 7.7. We only have to prove the uniquenece of the decompositions into the positive and negative parts; the rest follows from Theorems 5.3 and 7.4. Consider a non-degenerate torsion Hermitian form $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ and a decomposition $\mathcal{X} = \mathcal{X}_+ \perp \mathcal{X}_-$ given by Theorem 5.3. Suppose that $i : \mathcal{Y} \to \mathcal{X}$ is a monomorphism in $\mathcal{T}(\mathcal{C})$ such that the induced form $\phi|_\mathcal{Y}$ on $\mathcal{Y}$ (i.e. the form $\mathfrak{e}(i)\circ\phi\circ i : \mathcal{Y} \to \mathfrak{e}(\mathcal{Y})$) is non-degenerate and positively definite. The inclusion i followed by the canonical projections $\pi_\pm : \mathcal{X} \to \mathcal{X}_\pm$, determines the morphisms $f_\pm = \pi_\pm \circ i : \mathcal{Y} \to \mathcal{X}_\pm$. Denote by $\psi_\pm$ the forms on $\mathcal{X}$ induced on $\mathcal{Y}$ by the morphisms $f_\pm$. We have

$$\phi|_\mathcal{Y} = \psi_+ + \psi_-.$$

We know that $\phi|_\mathcal{Y}$ is non-degenerate and positively definite. Also we know that $-\psi_-$ is positively definite. Applying Lemma 7.8, we obtain that $\psi_+ = \phi|_\mathcal{Y} + (-\psi_-)$ is non-degenerate. Therefore the morphism $f_+ : \mathcal{Y} \to \mathcal{X}_+$ is a monomorphism in $\mathcal{T}(\mathcal{C})$.

It follows that if $\mathcal{X} = \mathcal{X}'_+ \perp \mathcal{X}'_-$ is another representation (4-46) then $\mathcal{X}'_+$ can be imbeded into $\mathcal{X}_+$ and similarly $\mathcal{X}_+$ can be imbeded into $\mathcal{X}'_+$. Now, we obtain that $\mathcal{X}_+$ is isomorphic to $\mathcal{X}'_+$, because of superfiniteness of $\mathcal{C}$. $\square$

7.10. Theorem. *Let $\mathcal{C}$ be a superfinite von Neumann category. Let $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ in $\mathcal{T}(\mathcal{C})$ be a non-degenerate torsion Hermitian form in $\mathcal{T}(\mathcal{C})$. Then ϕ is hyperbolic if and only if the torsion objects $\mathcal{X}_+$ and $\mathcal{X}_-$ are isomorphic in $\mathcal{T}(\mathcal{C})$.*

Proof. Suppose first that we know that $\mathcal{X}_+ \simeq \mathcal{X}_-$ in the decomposition of the form ϕ into an orthogonal sum of a positive and a negative forms. Then we may identify $\mathcal{X}_+ = \mathcal{X}_- = \mathcal{Y}$ and using Theorem 7.4 we may assume that $\phi = \psi \perp (-\psi)$, where $\psi : \mathcal{Y} \to \mathfrak{e}(\mathcal{Y})$ is a non-degenerate positive form on $\mathcal{Y}$. Then we have the following metabolizer $i : \mathcal{Y} \to \mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}$, where $i = i_1 + i_2$, the sum of two inclusions. This metabolizer is clearly a direct summand. Thus, ϕ is hyperbolic.

Now we want to prove the converse statement. Suppose that $\phi : \mathcal{X} \to \mathfrak{e}(\mathcal{X})$ is Hermitian and non-degenerate and let $\mathcal{Y} \subset \mathcal{X}$ be a metabolizer, $\mathcal{Y} = \mathcal{Y}^\perp$, in $\mathcal{X}$ which is a direct summand, $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$. ϕ is a morphism $\mathcal{Y} \oplus \mathcal{Z} \to \mathfrak{e}(\mathcal{Y}) \oplus \mathfrak{e}(\mathcal{Z})$, which is an isomorphism, and we know that the induced morphism $\mathcal{Y} \to \mathfrak{e}(\mathcal{Y})$ vanishes. Therefore we obtain:

(1) the induced morphism $\alpha : \mathcal{Y} \to \mathfrak{e}(\mathcal{Z})$ is a monomorphism (since its kernel would be a part of the kernel of ϕ);
(2) the other induced morphism $\alpha^\dagger : \mathcal{Z} \to \mathfrak{e}(\mathcal{Y})$ (cf. notation introduced in 1.3) is also a monomorphism (non-triviality of its kernel would contradict the condition $\mathcal{Y} = \mathcal{Y}^\perp$).

Since any torsion object is isomorphic to its dual (not canonically), we obtain that $\mathcal{Y}$ can be imbeded into $\mathcal{Z}$ and conversely, $\mathcal{Z}$ can be imbeded into $\mathcal{Y}$. Using superfiniteness of $\mathcal{C}$, we obtain that $\mathcal{Y}$ and $\mathcal{Z}$ are isomorphic and, moreover, the morphisms $\alpha : \mathcal{Y} \to \mathfrak{e}(\mathcal{Z})$ and $\alpha^\dagger : \mathcal{Z} \to \mathfrak{e}(\mathcal{Y})$ induced by ϕ, are isomorphisms.

Denote by $\gamma : \mathcal{Z} \to \mathfrak{e}(\mathcal{Z})$ the morphism induced by ϕ. Note that, we may assume without loss of generality, that $\gamma = 0$. If this condition is not satisfied, we may choose another imbeding $j' : \mathcal{Z} \to \mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$, where $j' = j + (\alpha^\dagger)^{-1}\gamma$ (here $j : \mathcal{Z} \to \mathcal{X}$ denotes the original imbeding), to achieve $\gamma = 0$.

Now we may identify $\mathcal{X}$ with $\mathcal{Y} \oplus \mathcal{Y}$ and assume that the form ϕ is given by the matrix

$$\begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$$

where $\beta : \mathcal{Y} \to \mathfrak{e}(\mathcal{Y})$ is a non-degenerate positive form on $\mathcal{Y}$. It is obvious that we may find a decomposition $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ into orthogonal sum of a positive and negative forms, where X_+ is $\mathcal{Y}$, imbeded into $\mathcal{X}$ via $i_1 + i_2$, and $\mathcal{X}_-$ is $\mathcal{Y}$, which is imbeded into $\mathcal{X}$ via $i_1 - i_2$. The result now follows from Theorem 7.7. $\square$

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